The *Nature* of Mathematics

13th Edition

KARL J. SMITH

Getting Help with the material in this text.

- Important terms are in **boldface** and are listed at the end of each chapter and in the **glossary.**
- Important ideas are reviewed at the end of each chapter.
- Types of problems are listed at the end of each chapter.
- The *Student's Survival Manual* lists the new terms for each chapter and enumerates the types of problems in each chapter.
- I also use this special font to speak to you directly out of the context of the regular textual material. I call these author's notes, and they are comments that I might say to you if we were chatting in my office about the content in this text.
- Road signs are used to help you with your journey through the text:

This stop sign means that you should stop and pay attention to this idea, since it will be used as you travel through the rest of the text.

Caution means that you will need to proceed more slowly to understand this material.

A bump symbol means to watch out, because you are coming to some difficult material.

This logo means you should check this out on the Web. The website for this text is **www.mathnature.com.** You will find homework hints, essential ideas, search engines, projects, and links to related topics.

Free mathematics tutorials, problems, and worksheets (with applets). This is an invaluable link:

http://www.analyzemath.com/

Webmath: Instant solution to your math problems:

http://www.webmath.com/

Ask Dr. Math is a question and answer service for K-12 math students and their teachers. A searchable archive is available by level and topic, together with frequently asked questions (FAQ).

http://mathforum.org/dr.math/

Online Math Calculators and Solvers. Easy-to-use online math calculators and solvers are available for various mathematics topics. These may be used to check homework solutions, as well as to practice on your own.

http://www.analyzemath.com/Calculators.html

Materials available on the Mathematics Archives are classified into five main categories: Topics in Mathematics, Software, Teaching Materials, Other Math Archives Features, and Other Links. This is a very useful site.

http://archives.math.utk.edu/

The Math Forum@Drexel is a great source of mathematical information that offers Math Talk, Math Problems and Puzzles, Math Help, and other Math Resources and Tools:

http://mathforum.org/

You can contact the author at: smithkjs@mathnature.com.

Thirteenth Edition

THE Nature of Mathematics

KARL J. SMITH

Santa Rosa Junior College

CENGAGE Learning®

Australia • Brazil • Mexico • Singapore • United Kingdom • United States

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*Optional sections.

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11 THE NATURE OF FINANCIAL MANAGEMENT 497

12 THE NATURE OF COUNTING 565

*Optional sections.

*Optional sections.

*Optional chapter.

I dedicate this book, with love,
to my wife, Linda.

Preface

Like almost every subject of human interest, mathematics is as easy or as difficult as we choose to make it. At the beginning of Chapter 1, I have included a Fable, and have addressed it directly to you, the student. I hope you will take the time to read it, and then ponder why I call it a fable.

You will notice street sign symbols used throughout this text. I use this stop sign to mean that you should stop and pay attention to this idea, since it will be used as you travel through the rest of the text.

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I also use this special font to speak to you directly out of the context of the regular textual material. I call these **author's notes;** they are comments that I might say to you if we were chatting in my office about the content in this text.

I frequently encounter people who tell me about their unpleasant experiences with mathematics. I have a true sympathy for those people, and I recall one of my elementary school teachers who assigned additional arithmetic problems as punishment. This can only create negative attitudes toward mathematics, which is indeed unfortunate. If elementary school teachers and parents have positive attitudes toward mathematics, their children cannot help but see some of the beauty of the subject. I want students to come away from this course with the feeling that mathematics can be pleasant, useful, and practical—and enjoyed for its own sake.

Since the first edition, my goal has been, and continues to be, to create a positive attitude toward mathematics. But the world, the students, and the professors are very different today than they were when I began writing this text. This is a very different text from its first printing, and this edition is very different from the previous edition. The world of knowledge is more accessible today (via the Internet) than at any time in history. Supplementary help is available on the Internet, and can be accessed at the following Web address: **www.mathnature.com**

All of the Web addresses mentioned in this text are linked to the above Web address. If you have access to a computer and the Internet, check out this Web address. You will find links to several search engines, history, and reference topics. You will find, for each section, homework hints, and a listing of essential ideas, projects, and links to related information on the Web.

This text was written for students who need a mathematics course to satisfy the general university competency requirement in mathematics. Because of the university requirement, many students enrolling in a course that uses my text have postponed taking this course as long as possible. They dread the experience, and come to class with a great deal of anxiety. Rather than simply presenting the technical details needed to proceed to the next course, I have attempted to give insight into what mathematics is, what it accomplishes, and how it is pursued as a human enterprise. However, at the same time, in this thirteenth edition I have included a great deal of material to help students estimate, calculate, and solve problems *outside* the classroom or textbook setting.

This text was written to meet the needs of all students and schools. How did I accomplish that goal? First, the chapters are almost independent of one another, and can be covered in any order appropriate to a particular audience. For example, in this edition, you will see the order of the chapters on measurement and on networks has been reversed from the previous edition. This shift was made in response to many of you who had been teaching these chapters for years.

Second, the problems are designed as the core of the course. There are problems that every student will find easy and this will provide the opportunity for success; there are also problems that are very challenging. Much interesting material appears in the problems, and students should get into the habit of reading (not necessarily working) all the problems whether or not they are assigned.

Level 1: Mechanical or drill problems

Level 2: Problems that require understanding of the concepts

Level 3: Problems that require problem-solving skills or original thinking

What Are the Major Themes of This Text?

The major themes of this text are *problem solving* and estimation in the context of presenting the great ideas in the history of mathematics.

I believe that *learning to solve problems is the principal reason for studying mathematics.* Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in most textbooks is one form of problem solving, but students also should be faced with non-text-type problems. In the first section of this edition, I introduce students to Pólya's problem-solving techniques, and these techniques are used throughout the text to solve non-text-type problems. Problem-solving examples are found throughout the text (marked as **PÓLYA'S METHOD** examples).

YinYang/Getty Images

Example 2 Retirement calculation Pólya's Method

Suppose you are 21 years old and will make monthly deposits to a bank account paying 4% annual interest compounded monthly.

***Option I*:** Pay yourself \$200 per month for 5 years and then leave the balance in the bank until age 65. (Total amount of deposits is $\$200 \times 5 \times 12 = \$12{,}000$.)

***Option II*:** Wait until you are 50 years old (the age most of us start thinking seriously about retirement) and then deposit \$200 per month until age 65. (Total amount of deposits is $\$200 \times 15 \times 12 = \$36{,}000$.)

Compare the amounts you would have from each of these options.

Solution We use Pólya's problem-solving guidelines for this example.

Understand the Problem. When most of us are 21 years old, we do not think about retirement. However, if we do, the results can be dramatic. With this example, we investigate the differences if we save early (for 5 years), or later (for 15 years).

Devise a Plan. We calculate the value of the annuity for the 5 years of the first option, and then calculate the effect of leaving the value at the end of 5 years (the annuity) in a savings account until retirement. This part of the problem is a future value problem because it becomes a lump-sum problem when deposits are no longer made (after 5 years). For the second option, we calculate the value of the annuity for 15 years.

Carry Out the Plan.

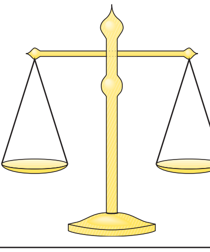
***Option I*:** \$200 per month for 5 years at 4% annual interest is an annuity with $m = 200$, $r = 0.04$, $t = 5$, and $n = 12$; this is an *annuity*.

$$A = 200\left[\frac{\left(1 + \frac{0.04}{12}\right)^{12(5)} - 1}{\frac{0.04}{12}}\right] \approx 13{,}259.80$$

You will find problems in *each* section that require Pólya's method for problem solving, and then you can practice your problem-solving skills with problems that are marked **Level 3, Problem Solving.**

Problem Solving Level 3

57. You have 9 coins, but you are told that one of the coins is counterfeit and weighs just a little more than an authentic coin. How can you determine the counterfeit with 2 weighings on a two-pan balance scale? (This problem is discussed in Chapter 3.)



Students should learn the language and notation of mathematics. Most students who have trouble with mathematics do not realize that mathematics *does require hard work.* The usual pattern for most mathematics students is to open the text to the assigned page of problems, and begin working. Only after getting "stuck" is an attempt made to "find it in the text." The final resort is reading the text. In this text, students are asked not only to "do math problems," but also to "experience mathematics." This means it is necessary to become involved with the **concepts** being presented, not "just get answers." In fact, the advertising slogan "Math Is Not a Spectator Sport" is an invitation that suggests that the only way to succeed in mathematics is to become involved with it.

MATH is not a spectator sport — BROOKS/COLE PUBLISHING COMPANY, PACIFIC GROVE, CALIFORNIA

Students will learn to receive mathematical ideas through listening, reading, and visualizing. They are expected to present mathematical ideas by speaking, writing, drawing pictures and graphs, and demonstrating with concrete models. There is a category of problems in each section that is designated **IN YOUR OWN WORDS,** and that provides practice in communication skills.

1. **IN YOUR OWN WORDS** Discuss the difference between solving word problems in textbooks and problem solving outside the classroom.
2. **IN YOUR OWN WORDS** In Problem Set 1.1, we asked you to discuss Pólya's problem-solving model. Now that you have spent some time reading this text, we ask the same question again to see whether your perspective has changed at all. Discuss Pólya's problem-solving model.

Students should view mathematics in historical perspective. There is no argument that mathematics has been a driving force in the history of civilization. In order to bring students closer to this history, I've included not only Historical Notes, but also a category of problems called **HISTORICAL QUEST**.

Historical Note

Karl Smith Library

Karl Gauss (1777–1855)

Along with Archimedes and Isaac Newton, Gauss is considered one of the three greatest mathematicians of all time. When he was 3 years old, he corrected an error in his father's payroll calculations. By the time he was 21, he had contributed more to mathematics than most do in a lifetime.

56. **HISTORICAL QUEST** The Historical Note on page 218 introduces the great mathematician Karl Gauss. Gauss kept a scientific diary containing 146 entries, some of which were independently discovered and published by others. On July 10, 1796, he wrote

What do you think this meant? Illustrate with some numerical examples.

Students should learn to think critically. Many colleges have a broad educational goal of increasing critical thinking skills. Wikipedia defines **critical thinking** as "purposeful and reflective judgment about what to believe or do in response to observations, experience, verbal or written expressions or arguments." Critical thinking might involve determining the meaning and significance of what is observed or expressed, or, concerning a given inference or argument, determining whether there is adequate justification to accept the conclusion as true. Critical thinking begins in earnest in Section 1.1 when we introduce Pólya's problem-solving method. These Pólya examples found throughout the text are not the usual "follow-the-leader"-type problems, but attempt, slowly, but surely, to teach critical thinking. The Problem Solving problems in almost every section continue this theme. The following sections are especially appropriate to teaching critical thinking skills: Problem Solving (1.1), Problem Solving Using Logic (3.5), Cryptography (5.8), Modeling Uncategorized Problems (6.9), Summary of Financial Formulas (11.7), Probability Models (13.3), Voting Dilemmas (17.2), Apportionment Paradoxes (17.4), and What Is Calculus? (18.1).

A Note for Instructors

The prerequisites for this course vary considerably, as do the backgrounds of students. Some schools have no prerequisites, while other schools have an intermediate algebra prerequisite. The students, as well, have heterogeneous backgrounds. Some have little or no mathematics skills; others have had a great deal of mathematics. Even though the usual prerequisite for using this text is intermediate algebra, a careful selection of topics and chapters would allow a class with a beginning algebra prerequisite to study the material effectively.

Feel free to arrange the material in a different order from that presented in the text. I have written the chapters to be as independent of one another as possible. There is much more material than could be covered in a single course. This text can be used in classes designed for liberal arts, teacher training, finite mathematics, college algebra, or a combination of these.

Over the years, many instructors from all over the country have told me that they love the material, love to teach from this text, but complain that there is just too much material in this text to cover in one, or even two, semesters. In response to these requests, I have divided some of the material into two separate volumes:

The Nature of Problem Solving in Geometry and Probability
The Nature of Problem Solving in Algebra

The first volume, *The Nature of Geometry and Probability,* includes Chapters 1, 2, 3, 7, 8, 9, 11, 12, and 13 from this text.

The second volume, *The Nature of Algebra,* includes Chapters 1, 4, 5, 6, 8, 10, 14, 15, 16, and 17 from this text.

Since the first edition of this text, I have attempted to make the chapters as independent as possible to allow instructors to "pick and choose" the chapters to custom design the course. It is possible to customize your own text for class. Details are available from your sales representative.

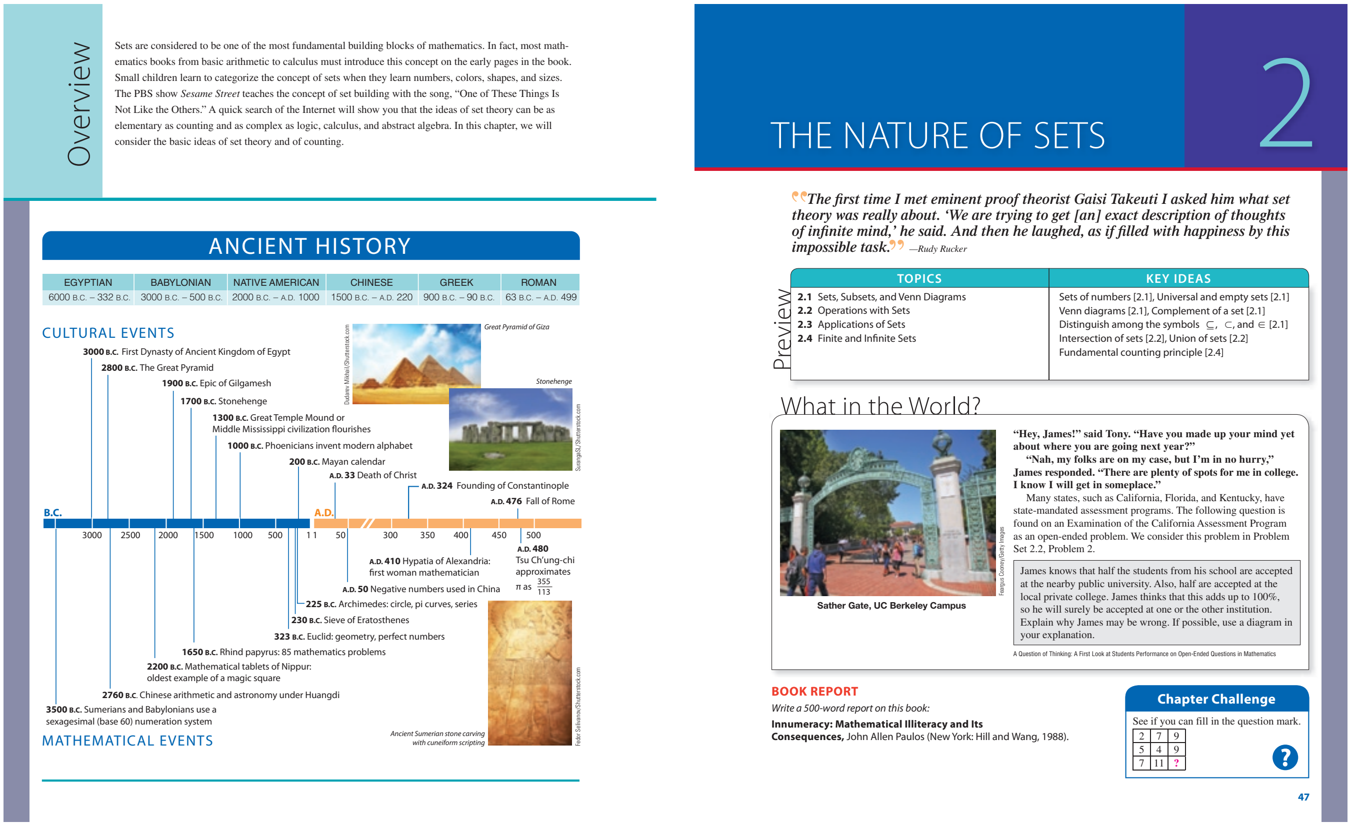
One of the advantages of using a textbook that has traveled through many editions is that it is well seasoned. Errors are minimal, pedagogy is excellent, and it is easy to use; in other words, it works. For example, you will find that the sections and chapters are about the right length . . . each section will take about one classroom day. The problem sets are graded so that you can teach the course at different levels of difficulty, depending on the assigned problems. The problem sets are uniform in length (60 problems each), which facilitates the assigning of problems from day to day. The chapter reviews are complete and lead students to the type of review they will need to prepare for an examination.

Changes from the Previous Edition

As a result of extensive reviewer feedback, there are many new ideas and changes in this edition.

- Mathematical history has been an integral feature of this text since its inception, and I have long used Historical Notes to bring the human story into our venture through this text. In the last edition, I experimented with a new type of problem called an **HISTORICAL QUEST**, and it has proved to be an overwhelming success, so I have greatly expanded its use in this edition. These problems are designed to *involve* the student in the historical development of the great ideas in mathematical history.

- Added a new feature called **FAST FACTS** to provide some essential information at exactly the right time and place.
- The chapter openings have been completely redesigned into two-page openers.



Overview

Sets are considered to be one of the most fundamental building blocks of mathematics. In fact, most mathematics books from basic arithmetic to calculus must introduce this concept on the early pages in the book. Small children learn to categorize the concept of sets when they learn numbers, colors, shapes, and sizes. The PBS show *Sesame Street* teaches the concept of set building with the song, "One of These Things Is Not Like the Others." A quick search of the Internet will show you that the ideas of set theory can be as elementary as counting and as complex as logic, calculus, and abstract algebra. In this chapter, we will consider the basic ideas of set theory and of counting.

ANCIENT HISTORY

EGYPTIAN	BABYLONIAN	NATIVE AMERICAN	CHINESE	GREEK	ROMAN
6000 B.C. – 332 B.C.	3000 B.C. – 500 B.C.	2000 B.C. – A.D. 1000	1500 B.C. – A.D. 220	900 B.C. – 90 B.C.	63 B.C. – A.D. 499

CULTURAL EVENTS

3000 B.C. First Dynasty of Ancient Kingdom of Egypt
2800 B.C. The Great Pyramid
1900 B.C. Epic of Gilgamesh
1700 B.C. Stonehenge
1300 B.C. Great Temple Mound or Middle Mississippi civilization flourishes
1000 B.C. Phoenicians invent modern alphabet
200 B.C. Mayan calendar
A.D. 33 Death of Christ
A.D. 324 Founding of Constantinople
A.D. 476 Fall of Rome

Great Pyramid of Giza
Stonehenge

B.C. 3000 2500 2000 1500 1000 500 1 1 50 A.D. 300 350 400 450 500

A.D. 480 Tsu Ch'ung-chi approximates π as $\frac{355}{113}$
A.D. 410 Hypatia of Alexandria: first woman mathematician
A.D. 50 Negative numbers used in China
225 B.C. Archimedes: circle, pi curves, series
230 B.C. Sieve of Eratosthenes
323 B.C. Euclid: geometry, perfect numbers
1650 B.C. Rhind papyrus: 85 mathematics problems
2200 B.C. Mathematical tablets of Nippur: oldest example of a magic square
2760 B.C. Chinese arithmetic and astronomy under Huangdi
3500 B.C. Sumerians and Babylonians use a sexagesimal (base 60) numeration system

MATHEMATICAL EVENTS

Ancient Sumerian stone carving with cuneiform scripting

THE NATURE OF SETS 2

"The first time I met eminent proof theorist Gaisi Takeuti I asked him what set theory was really about. 'We are trying to get [an] exact description of thoughts of infinite mind,' he said. And then he laughed, as if filled with happiness by this impossible task." —Rudy Rucker

Preview

TOPICS	KEY IDEAS
2.1 Sets, Subsets, and Venn Diagrams 2.2 Operations with Sets 2.3 Applications of Sets 2.4 Finite and Infinite Sets	Sets of numbers [2.1], Universal and empty sets [2.1] Venn diagrams [2.1], Complement of a set [2.1] Distinguish among the symbols ⊆, ⊂, and ∈ [2.1] Intersection of sets [2.2], Union of sets [2.2] Fundamental counting principle [2.4]

What in the World?

Sather Gate, UC Berkeley Campus

"Hey, James!" said Tony. "Have you made up your mind yet about where you are going next year?"

"Nah, my folks are on my case, but I'm in no hurry," James responded. "There are plenty of spots for me in college. I know I will get in someplace."

Many states, such as California, Florida, and Kentucky, have state-mandated assessment programs. The following question is found on an Examination of the California Assessment Program as an open-ended problem. We consider this problem in Problem Set 2.2, Problem 2.

James knows that half the students from his school are accepted at the nearby public university. Also, half are accepted at the local private college. James thinks that this adds up to 100%, so he will surely be accepted at one or the other institution. Explain why James may be wrong. If possible, use a diagram in your explanation.

A Question of Thinking: A First Look at Students Performance on Open-Ended Questions in Mathematics

BOOK REPORT

Write a 500-word report on this book:

Innumeracy: Mathematical Illiteracy and Its Consequences, John Allen Paulos (New York: Hill and Wang, 1988).

Chapter Challenge

See if you can fill in the question mark.

2	7	9
5	4	9
7	11	?

47

- Reversed the order of Chapters 8 and 9, so that The Nature of Measurement now follows directly behind The Nature of Geometry, since those two chapters are closely linked.
- Added a section on stochastic processes and tree diagrams.
- Added a new section on Bayes' theorem.
- Added Laffer curves to Section 15.4.
- Reworked the group research projects; added 5 new substantial group projects.
- Reworked the individual research projects; added 40 new projects.
- Reworked and updated the problem sets; all problem sets now have 60 problems. The problem sets are uniform in length (60 problems each), which facilitates the assigning of problems from day to day. (For example, 3-60, multiples of 3 work well with the paired nature of the problems.)
- Over 700 problems have been added or revised.
- The prologue and epilogue have been redesigned and rewritten to offer unique "bookends" to the material in the text. The prologue asks the question, "**Why Math?**". This prologue not only puts mathematics into a historical perspective, but also is designed to help students to begin thinking about problem solving. The problems accompanying this prologue could serve as a pre-test or diagnostic test, but I use these prologue problems to let the students know that this text will not be like other math books they may have used in the past. The epilogue, "**Why Not Math?**", is designed to tie together many parts of the text (which may or may not have been "covered" in the class) to show that there are many rooms in the mansion known as mathematics. The problems accompanying this epilogue could serve as a review to show that it would be difficult to choose a course of study in college without somehow being touched by mathematics. When have you seen a mathematics textbook that asks the question "Why study mathematics?" and then actually produces an example to show it?*

*See Example 2, Section 11.5.

Help for the Instructor

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- Promote student success through interactivity, multimedia, and exercises

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Annotated Instructor's Edition (978-1-305-65778-6)

This instructor's version of the complete student text includes the answers to each exercise printed next to the respective exercise.

Instructor's Manual (978-1-305-65777-9)

Karl Smith has written an extensive *Instructor's Manual* (over 500 pages) to accompany this text. It includes the complete solutions to all the problems (including the "Problem Solving" problems), as well as teaching suggestions and transparency masters. These transparencies are included (in color) as a PDF file at **www.mathnature.com** so that you can download them for use in your classroom.

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Help for the Student

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MindTap for Mathematics is a digital-learning solution that places learning at the center of the experience. In addition to algorithmically generated problems, immediate feedback, and a powerful answer evaluation and grading system, MindTap for Mathematics gives you a personalized path of dynamic assignments, a focused improvement plan, and just-in-time, integrated remediation that turns cookie cutter into cutting edge, apathy into engagement, and memorizers into higher-level thinkers.

Learn more at **www.cengage.com/mindtap**.

Student Survival and Solutions Manual (978-1-305-65776-2)

Authored by Karl Smith, this manual is intended to help students with their success. It contains the complete solution of approximately one-half of the problems in the text. Also available are sample tests, not only in hard copy form, but also in electronic form for both Windows and Macintosh formats.

Text Companion Website at www.mathnature.com

Created and updated by Karl Smith, the website offers supplementary help and practice for students. All of the Web addresses mentioned in the text are linked to the above Web address. You will find links to several search engines, history, and reference topics. You will find, for each section, homework hints, and a listing of essential ideas, projects, and links to related information on the Web.

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I'd especially like to thank Arian Hojat for his input about the great pioneers of Persian/ Iranian heritage and their contribution to human civilization.

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Prologue: Why Math?

A HISTORICAL OVERVIEW

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is." —John van Neumann

Whether you love or loathe mathematics, it is hard to deny its importance in the development of the main ideas of this world! Read the BON VOYAGE invitation in the Overview to Chapter 1. The goal of this text is to help you to discover an answer to the question, "Why study math?"

The study of mathematics can be organized as a history or story of the development of mathematical ideas, or it can be organized by topic. The intended audience of this text dictates that the development should be by topic, but mathematics involves real people with real stories, so you will find this text to be very historical in its presentation. This overview rearranges the material you will encounter in the text into a historical timeline. It is not intended to be read as a history of mathematics, but rather as an overview to make you want to do further investigation. Sit back, relax, and use this overview as a *starting place* to expand your knowledge about the beginnings of some of the greatest ideas in the history of the world!

We have divided this history of mathematics into five chronological periods:

Ancient History	Chapter 2 overview	6000 B.C. to A.D. 499
Hindu and Persian Period	Chapter 6 overview	500 to 1199
Transition Period	Chapter 10 overview	1200 to 1599
Age of Reason	Chapter 14 overview	1600 to 1799
Modern Period	Chapter 18 overview	1800 to present

Ancient History: 6000 B.C. to A.D. 499

The ancient period includes the Egyptian (6000 B.C.–332 B.C.), Babylonian (3000 B.C.–500 B.C.), Native American (2000 B.C.–A.D. 1000), Chinese (1500 B.C.–A.D. 220), Greek (900 B.C.–90 B.C.), and Roman (63 B.C.–A.D. 499) civilizations.


Gianni Dagli Orti/Fine Art/Corbis

Sumerian clay tablet

Mesopotamia is an ancient region located in southwest Asia between the lower Tigris and Euphrates rivers and is historically known as the birthplace of civilization. It is part of modern Iraq. Mesopotamian mathematics refers to the mathematics of the ancient Babylonians, and this mathematics is sometimes referred to as Sumerian mathematics. Over 50,000 tablets from Mesopotamia have been found and are exhibited at major museums around the world.

Interesting readings about Babylon can be found in a book on the history of mathematics, such as *An Introduction to the History of Mathematics,* 6th edition, by Howard Eves (New York: Saunders, 1990), or by looking at the many sources on the World Wide Web. You can find links to such websites, as well as all websites in this text, by looking at the Web page for this text:

www.mathnature.com

This Web page allows you to access a world of information by using the links provided.

The mathematics of this period was very practical, and it was used in construction, surveying, record keeping, and the creation of calendars. The culture of the Babylonians reached its height about 2500 B.C., and about 1700 B.C. King Hammurabi formulated a famous code of law. In 330 B.C., Alexander the Great conquered Asia Minor, ending the great Persian (Achaemenid) Empire. Even though there was a great deal of political and social upheaval during this period, there was a continuity in the development of mathematics from ancient times to the time of Alexander.

The main information we have about the civilization and mathematics of the Babylonians is their numeration system, which we introduce in Section 4.1 of this text. The Babylonian numeration system was positional with base 60. It did not have a 0 symbol, but it did represent fractions, squares, square roots, cubes, and cube roots. We have evidence that the Babylonians knew the quadratic formula and that they had stated algebraic problems verbally. The base 60 system of the Babylonians led to the division of a circle into 360 equal parts that today we call degrees, and each degree was in turn divided into 360 parts that today we call seconds. The Greek astronomer Ptolemy (A.D. 85–165) used this Babylonian system, which no doubt is why we have minutes, seconds, and degree measurement today.

The Egyptian civilization existed from about 4000 B.C., and was less influenced by foreign powers than was the Babylonian civilization. Egypt was divided into two kingdoms

Bettmann/CORBIS

Egyptian hieroglyphics: Inscription and relief from the grave of Prince Rahdep (ca. 2800 B.C.)

until about 3000 B.C., when the ruler Menes unified Egypt and consequently became known as the founder of the first dynasty in 2500 B.C. This was the Egyptians' pyramid-building period, and the Great Pyramid of Cheops was built around 2600 B.C. (Chapter 7, page 362; see The Riddle of the Pyramids in the Historical Note in Problem Set 7.5).

The Egyptians developed their own pictorial way of writing, called *hieroglyphics,* and their numeration system was consequently very pictorial (Chapter 4).

The Egyptian numeration system is an example of a simple grouping system. Although the Egyptians were able to write fractions, they used only unit fractions. Like the Babylonians, they had not developed a symbol for zero. Since the writing of the Egyptians was on papyrus, and not on tablets as with the Babylonians, most of the written history has been lost. Our information comes from the Rhind papyrus, discovered in 1858 and dated to about 1700 B.C., and the Moscow papyrus, which has been dated to about the same time period.

The mathematics of the Egyptians remained remarkably unchanged from the time of the first dynasty to the time of Alexander the Great, who conquered Egypt in 332 B.C. The Egyptians did surveying using a unique method of stretching rope, so they referred to their surveyors as "rope stretchers." The basic unit used by the Egyptians for measuring length was the *cubit*, which was the distance from a person's elbow to the end of the middle finger. A *khet* was defined to equal 100 cubits; khets were used by the Egyptians when land was surveyed. The Egyptians did not have the concept of a variable, and all of their problems were verbal or arithmetic. Even though they solved many equations, they used the word *AHA* or *heap* in place of the variable. For an example of an Egyptian problem, see Ahmes' dilemma in Chapter 1 and the statement of the problem in terms of Thoth, an ancient Egyptian god of wisdom and learning.

The Egyptians had formulas for the area of a circle and the volume of a cube, box, cylinder, and other figures. Particularly remarkable is their formula for the volume of a truncated pyramid of a square base, which in modern notation is

$$V = \frac{h}{3}(a^2 + ab + b^2)$$

where h is the height and a and b are the sides of the top and bottom. Even though we are not certain the Egyptians knew of the Pythagorean theorem, we believe they did because the rope stretchers had knots on their ropes that would form right triangles. They had a very good reckoning of the calendar and knew that a solar year was approximately $365\frac{1}{4}$ days long. They chose as the first day of their year the day on which the Nile would flood.

Contemporaneous with the great civilizations in Mesopotamia was the great Mayan civilization in what is now Mexico. Just as with the Mesopotamian civilizations, the Olmeca and Mayan civilizations lay between two great rivers, in this case the Grijalva and Papaloapan rivers. Sometimes the Olmecas are referred to as the Tenocelome. The Olmeca culture is considered the mother culture of the Americas. What we know about the Olmecas centers around their art. We do know they were a farming community. The Mayan civilization began around 2600 B.C. and gave rise to the Olmecas. The Olmecas had developed a written hieroglyphic language by 700 B.C., and they had a very accurate solar calendar. The Mayan culture had developed a positional numeration system. You will find the influences from this period discussed throughout this text.

Greek mathematics began in 585 B.C. when Thales, the first of the Seven Sages of Greece (625–547 B.C.), traveled to Egypt.*

*The Seven Sages in Greek history refer to Thales of Miletus, Bias of Priene, Chilo of Sparta, Cleobulus of Rhodes, Periander of Corinth, Pittacus of Mitylene, and Solon of Athens; they were famous because of their practical knowledge about the world and how things work.

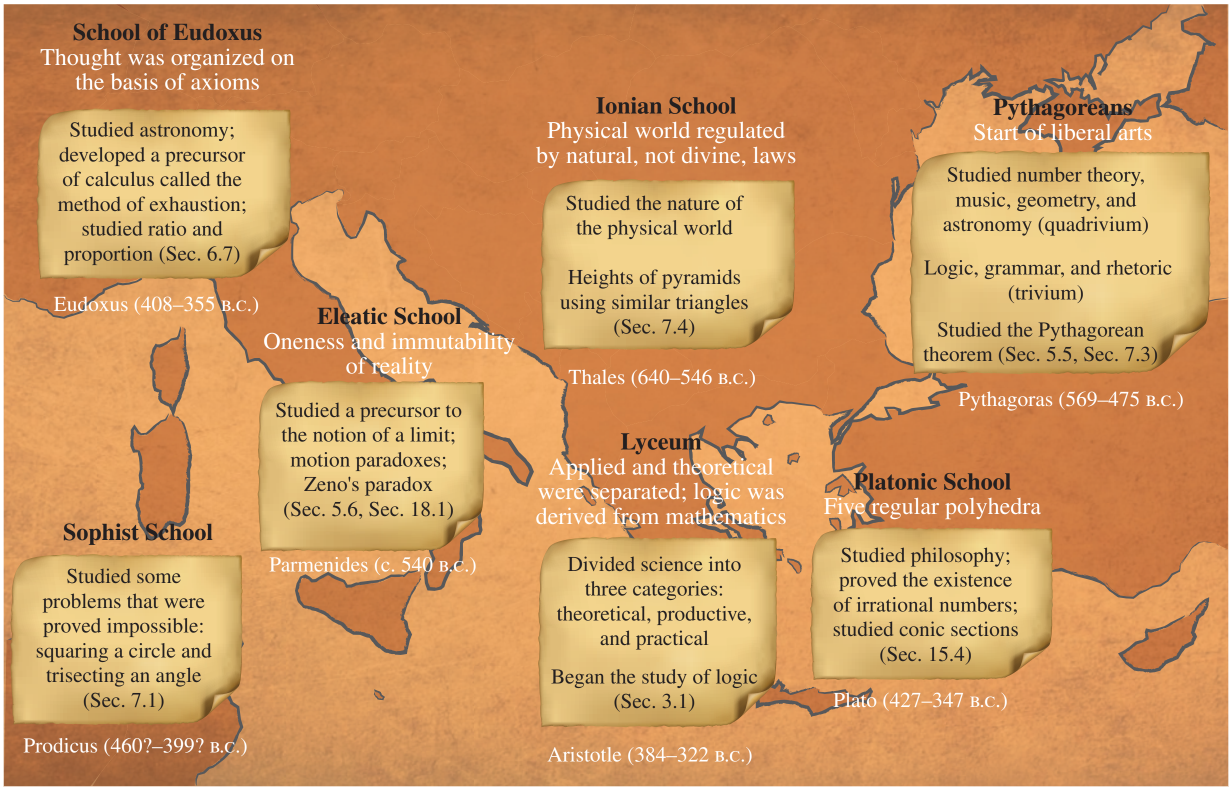

FIGURE 1 **Greek schools from 585 B.C. to 352 B.C.**

The Greek civilization was most influential in our history of mathematics. So striking was its influence that the historian Morris Kline declares, "One of the great problems of the history of civilization is how to account for the brilliance and creativity of the ancient Greeks."* The Greeks settled in Asia Minor, modern Greece, southern Italy, Sicily, Crete, and North Africa. They replaced the various hieroglyphic systems with the Phoenician alphabet, and with that they were able to become more literate and more capable of recording history and ideas. The Greeks had their own numeration system. They had fractions and some irrational numbers, including π.

The great mathematical contributions of the Greeks are Euclid's *Elements* and Apollonius' *Conic Sections* (page 735, Figure 15.28). Greek knowledge developed in several centers or schools. (See Figure 2 on page P4 for a depiction of one of these centers of learning.) The first was founded by Thales (ca. 640–546 B.C.) and known as the Ionian in Miletus. It is reported that while he was traveling and studying in Egypt, Thales calculated the heights of the pyramids by using similar triangles (see Section 7.4). You can read about these great Greek mathematicians in *Mathematical Thought from Ancient to Modern Times,* by Morris Kline.† You can also refer to the World Wide Web at **www.mathnature.com.**

Between 585 B.C. and 352 B.C., schools flourished and established the foundations for the way knowledge is organized today. Figure 1 shows each of the seven major schools, along with each school's most notable contribution. References to textual discussion are shown within each school of thought, along with the principal person for each of these schools. Books have been written about the importance of each of these Greek schools, and several links can be found at **www.mathnature.com.**

One of the three greatest mathematicians in the entire history of mathematics was Archimedes (287–212 B.C.). His accomplishments are truly remarkable, and you should seek out other sources about the magnitude of his accomplishments. He invented a pump (the Archimedean screw), military engines and weapons, and catapults; in addition, he used a parabolic mirror as a weapon by concentrating the sun's rays on the invading Roman ships. "The most famous of the stories about Archimedes is his discovery of the method of testing the debasement of a crown of gold. The king of Syracuse had ordered the crown. When it was delivered, he suspected that it was filled with baser metals and sent it to Archimedes to devise some method of testing the contents without, of course, destroying the workmanship. Archimedes pondered the problem; one day while bathing he observed that his body was partly buoyed up by the water and suddenly grasped the principle that enabled him to solve the problem. He was so excited by this discovery that he ran out into the street naked shouting, 'Eureka!' ('I have found it!').

*p. 24, *Mathematical Thought from Ancient to Modern Times* by Morris Kline (New York: Oxford University Press, 1972).
†New York: Oxford University Press, 1972.

FIGURE 2 ***The School at Athens*** **by Raphael, 1509. This fresco includes portraits of Raphael's contemporaries and demonstrates the use of perspective. Note the figures in the lower right, who are, no doubt, discussing mathematics.**

He had discovered that a body immersed in water is buoyed up by a force equal to the weight of the water displaced, and by means of this principle was able to determine the contents of the crown."*

The Romans conquered the world, but their mathematical contributions were minor. We introduce the Roman numerals in Section 4.1; Romans' fractions were based on a duodecimal (base 12) system and are still used today in certain circumstances. The unit of weight was the *as* and one-twelfth of this was the *uncia,* from which we get our measurements of *ounce* and *inch*, respectively.

The Romans improved on our calendar and set up the notion of leap year every four years. The Julian calendar was adopted in 45 B.C. The Romans conquered Greece and Mesopotamia, and in 47 B.C., they set fire to the Egyptian fleet in the harbor of Alexandria. The fire spread to the city and burned the library, destroying two and a half centuries of book-collecting, including all the important knowledge of the time.

Another great world civilization existed in China and also developed a decimal numeration system and used a decimal system with symbols 1, 2, 3,· · ·, 9, 10, 100, 1000, and 10,000.

Calculations were performed using small bamboo counting rods, which eventually evolved into the abacus. Our first historical reference to the Chinese culture is the yin-yang

*pp. 105–106, *Mathematical Thought from Ancient to Modern Times* by Morris Kline (New York: Oxford University Press, 1972).

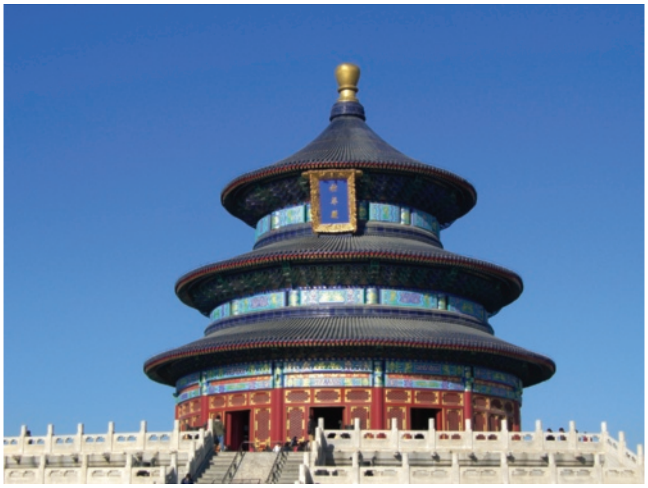

FIGURE 3 **Tiantan Temple, Beijing, China, constructed from A.D. 1406 to 1420**

symbol, which has its roots in ancient cosmology. The original meaning is representative of the mountains, both the bright side and the dark side. The "yin" represents the female, or shaded, aspect; the earth, the darkness, the moon, and passivity. The "yang" represents the male, light, sun, heaven, and the active principle in nature. These words can be traced back to the Shang and Chou dynasties (1550–1050 B.C.), but most scholars credit them to the Han Dynasty (220–206 B.C.). One of the first examples of a magic square comes from the Lo River around 200 B.C., where legend tells us that the emperor Yu of the Shang Dynasty received a magic square on the back of a tortoise's shell.

From 100 B.C. to A.D. 100, the Chinese described the motion of the planets, as well as what is the earliest known proof of the Pythagorean theorem. The longest surviving and most influential Chinese math book is dated from the beginning of the Han Dynasty around A.D. 50. It includes measurement and area problems, proportions, volumes, and some approximations for π. Sun Zi (ca. A.D. 250) wrote his mathematical manual, which included the "Chinese remainder problem": Find n so that upon division by 3 you obtain a remainder of 2; upon division by 5 a remainder of 3; and upon division by 7 you get a remainder of 2. His solution: Add 140, 63, 30 to obtain 233, and subtract 210 to obtain 23. Zhang Qiujian (ca. A.D. 450) wrote a mathematics manual that included a formula for summing the terms of an arithmetic sequence, along with the solution to a system of two linear equations in three unknowns. The problem is the "One Hundred Fowl Problem," and is included in Problem Set 5.7 (page 235). At the end of this historic period, the mathematician and astronomer Wang Xiaotong (ca. A.D. 626) solved cubic equations by generalization of an algorithm for finding the cube root.

Check **www.mathnature.com** for links to many excellent sites on Greek mathematics.

Hindu and Persian Period: 500 to 1199

Much of the mathematics that we read in contemporary mathematics textbooks ignores the rich history of this period. Included on the World Wide Web are some very good sources for this period. Check our website **www.mathnature.com** for some links. The Hindu civilization dates back to 2000 B.C., but the first recorded mathematics was during the Śulvasūtra period from 800 B.C. to 200 A.D. In the third century, Brahmi symbols were used for 1, 2, 3, ⋯, 9 and are significant because there was a single symbol for each number. There was no zero or positional notation at this time, but by A.D. 600 the Hindus used the Brahmi symbols with positional notation. In Chapter 4, we will discuss a numeration system that eventually evolved from these Brahmi symbols. For fractions, the Hindus used sexagesimal positional notation in astronomy, but in other applications they used a ratio of integers and wrote $\begin{smallmatrix}3\\4\end{smallmatrix}$ (without the fractional bar we use today). The first mathematically important period was the second period, A.D. 200–1200. The important mathematicians of this period are Āryabhata (A.D. 476–550), Brahmagupta (A.D. 598–670), Mahāvīra (9th century), and Bhāskara (1114–1185). In Chapter 6, we include some historical questions from Bhāskara and Brahmagupta.

The Hindus developed arithmetic independently of geometry and had a fairly good knowledge of rudimentary algebra. They knew that quadratic equations had two solutions, and they had a fairly good approximation for π. Astronomy motivated their study of trigonometry. Around 1200, scientific activity in India declined, and mathematical progress ceased and did not revive until the British conquered India in the 18th century.

The Persians invited Hindu scientists to settle in Baghdad, and when Plato's Academy closed in A.D. 529, many scholars traveled to Persia and became part of the Iranian tradition of science and mathematics. Omar Khayyám (1048–1122) and Nasîr-Eddin (1201–1274), both renowned Persian scholars, worked freely with irrationals, which contrasts with the Greek idea of number. What we call Pascal's triangle dates back to this period.

The word *algebra* comes from the Persians in a book by the Persian astronomer Mohammed ibn Musa al-Khwârizmî (780–850) entitled *Hisâb al-jabr w'al muqâbala.* Due to the Arab conquest of Persia, Persian scholars (notably Nasir-Eddin and al-Khwârizmî) were obliged to publish their works in the Arabic language and not Persian, causing many historians to falsely label the texts as products of Arab scholars. Al-Khwârizmî solved quadratic equations and knew there are two roots, and even though the Persians gave algebraic solutions of quadratic equations, they explained their work geometrically. They solved some cubics, but could solve only simple trigonometric problems.

Check **www.mathnature.com** for links to many excellent sites on Hindu and Arabian mathematics.

Lordprice Collection / Alamy

FIGURE 4 Omar Khayyám was a Persian mathematician, astronomer, philosopher, and poet. He is the symbol of pure love in the history of oriental wisdom.

Transition Period: 1200 to 1599

Mathematics during the Middle Ages was transitional between the great early civilizations and the Renaissance.

In the 1400s the Black Death killed over 70% of the European population. The Turks conquered Constantinople, and many Eastern scholars traveled to Europe, spreading Greek knowledge as they traveled. The period from 1400 to 1600, known as the Renaissance, forever changed the intellectual outlook in Europe and raised up mathematical thinking to new levels. Johann Gutenberg's invention of printing with movable type in 1450 changed the complexion of the world. Linen and cotton paper, which the Europeans learned about from the Chinese through the Arabians, came at precisely the right historical moment. The first printed edition of Euclid's *Elements* in a Latin translation appeared in 1482. Other early printed books were Apollonius' *Conic Sections*, Pappus' works, and Diophantus' *Arithmetica.*

The first breakthrough in mathematics was by artists who discovered mathematical perspective. The theoretical genius in mathematical perspective was Leone Alberti (1404–1472). He was a secretary in the Papal Chancery writing biographies of the saints, but his work *Della Pictura* on the laws of perspective (1435) was a masterpiece. He said, "Nothing pleases me so much as mathematical investigations and demonstrations, especially when I can turn them into some useful practice drawing from mathematics and the principles of painting perspective and some amazing propositions on the moving of weights." He collaborated with Toscanelli, who supplied Columbus with maps for his first voyage. The most famous mathematician among the Renaissance artists was Albrecht Dürer (1471–1528). The most significant development of the Renaissance was the breakthrough in astronomical theory by Nicolaus Copernicus (1473–1543) and Johannes Kepler (1571–1630). There were no really significant new results in mathematics during this period of history.

It is interesting to tie together some of the previous timelines to trace the history of algebra. It began around 2000 B.C. in Egypt and Babylon. This knowledge was incorporated into the mathematics of Greece between 500 B.C. and A.D. 320, as well as into the Persian civilization and Indian mathematics around A.D. 1000. By the Transition Period, the great ideas of algebra had made their way to Europe, as shown in Figure 6. Additional information can be found on the World Wide Web; check our Web page at **www.mathnature.com.**

Age of Reason: 1600 to 1799

From Shakespeare and Galileo to Peter the Great and the great Bernoulli family, this period, also called the Age of Genius, marks the growth of intellectual endeavors; both technology and knowledge grew as never seen before in history. A great deal of the content of this book focuses on discoveries from this period of time, so instead of providing a commentary in this overview, we will simply list the references to this period in world history. Other sources and links are found on our Web page at **www.mathnature.com.**

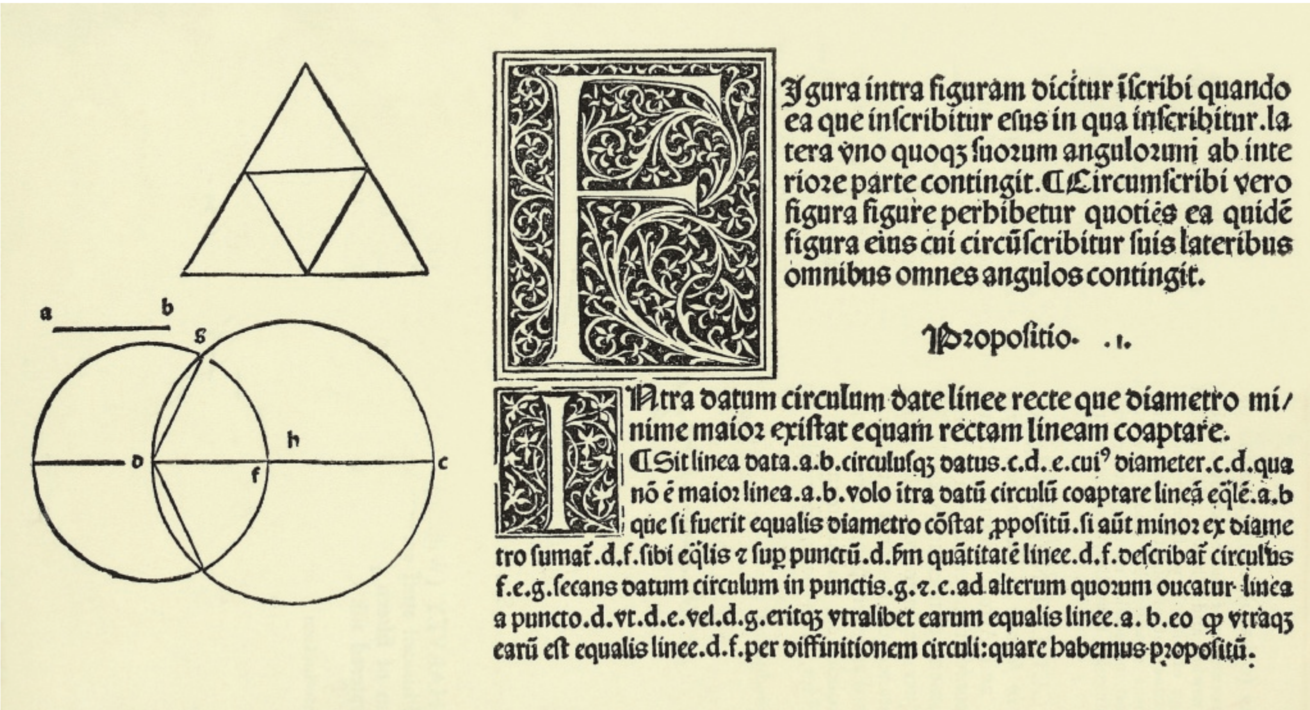

Figura intra figuram dicitur iscribi quando ea que inscribitur eius in qua inscribitur. latera vno quoqz suorum angulorum ab interiore parte contingit. ¶ Circumscribi vero figura figure perhibetur quoties ea quidẽ figura eius cui circũscribitur suis lateribus omnibus omnes angulos contingit.

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Bettmann/Corbis

FIGURE 5 **Euclid's *Elements* contained the cumulative knowledge of the ancient Greek mathematicians.**

Modern Period: 1800 to Present

This period marks the dawn of modern mathematics, and it includes all of the discoveries of the last two centuries. The Early Modern Period was characterized by experimentation and formalization of the ideas germinated in the previous century. There is so much that we could say about the period from 1700 to 1799. The mathematics that you studied in high school represents, for the most part, the ideas formulated during this period. Take a look at the mathematical events in the following timeline, and you will see an abundance of discoveries, often embodied in the contents of entire books. One of the best sources of information about this period is found at these websites:

www.mathnature.com and **http://www-history.mcs.st-and.ac.uk/~history/Indexes/Full_Chron.html**

Students often think that all the important mathematics has been done, and there is nothing new to be discovered, but this is not true. Mathematics is alive and constantly changing. This textbook was first published in 1973, and as this 13th edition is prepared in 2015, we are struck with all the significant mathematics that has been discovered during the life of this text:

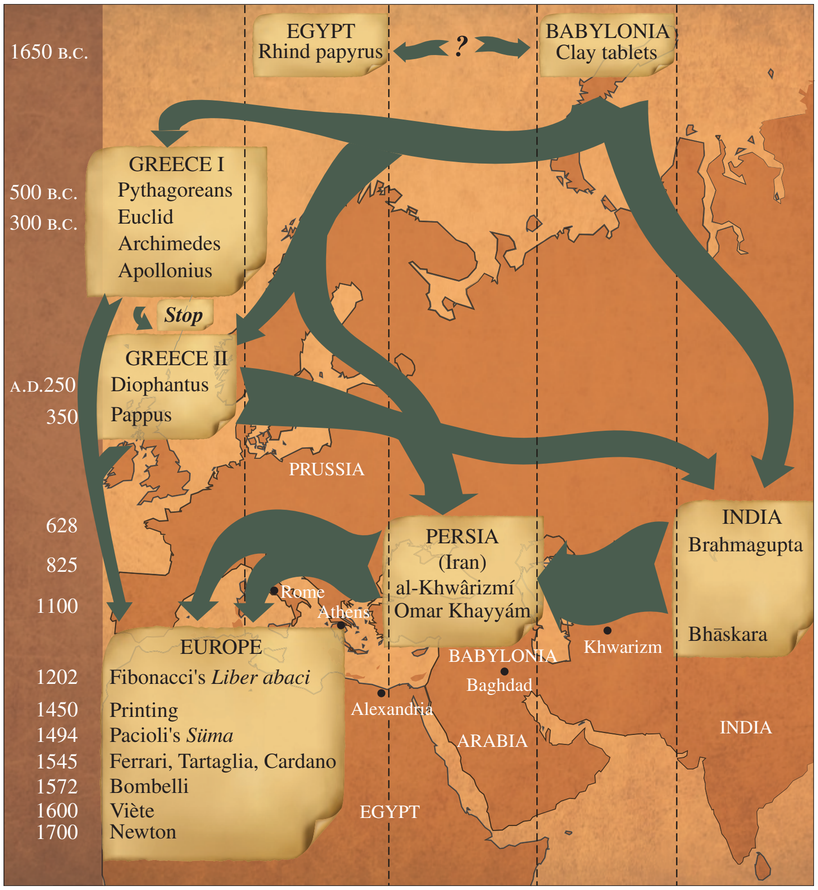


FIGURE 6 Mainstreams in the flow of algebra

Date	Topic	Where to Look in This Edition
1976	Appel and Haken solve the four-color problem.	Project 1.6 and Section 9.3
1977	Apple II personal computer introduced.	Section 4.5
1992	World Wide Web released by CREN.	Figure 4.13
1993	Andrew Wiles proves Fermat's last theorem.	Project 1.6 and Project G24
1994	RSA unbreakable code encryption invented.	Section 5.8
1998	Kepler's sphere-packing problem solved.	Project 1.6
2003	Poincaré conjecture proved by Grigori Perelman.	Section 9.3
2004	Mark Zuckerberg writes Facebook.	Section 4.5
2008	Edison Smith, George Woltman, et al. find largest known prime.	Table 5.3
2010	Grigori Perelman rejects the $1,000,000 Millennium Prize.	Section 9.3
2012	Shouryya Ray calculates the path of a projectile under gravity and subject to air resistance.	Epilogue

A quick search of the current mathematical journals shows a plethora of new topics and discoveries: algebraic geometry, category theory, combinatorics (Chapter 12), dynamical systems, homology, information theory, *k*-theory, logic, number theory, representation theory, symbolic geometry, and topology (Chapter 9). Some of these topics are included (as shown) in this edition, but the others are beyond the scope of this text.

There is no way a short commentary or overview can convey the richness or implications of the mathematical discoveries of this period. As we enter the new millennium, we can only imagine and dream about what is to come!

One of the major themes of this text is problem solving. The following problem set is a potpourri of problems that should give you a foretaste of the variety of ideas and concepts that we will consider in this text. Although none of these problems is to be considered routine, you might wish to attempt to work some of them before you begin, and then return to these problems at the end of your study in this text.

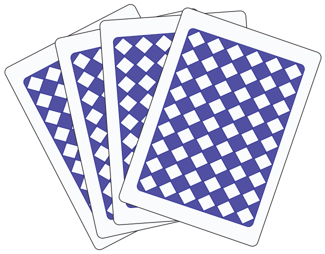
Prologue Problem Set

1. **IN YOUR OWN WORDS** The prologue provides a historical overview and asks the question, "Why Math?" This question seems appropriate at the beginning of a college mathematics course. Why do you think it is important to study mathematics?
2. **IN YOUR OWN WORDS** The epilogue for this text provides an essay of how the material of this text relates specifically to the natural sciences, social sciences, and humanities. In a sense, this is a recap of the question in Problem 1. In the prologue you are looking forward to the text and in the epilogue you are looking backward at what you did in the text. In the epilogue, we ask the question, "Why Not Math?" To answer this question, write a few paragraphs about why someone would not study mathematics.
3. **HISTORICAL QUEST** What are the five chronological historical periods into which the prologue is divided? Which period seems the most interesting to you, and why?
4. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Ancient Period.
5. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Hindu and Persian Period.
6. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Transition Period.
7. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Age of Reason.
8. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Modern Period.
9. **IN YOUR OWN WORDS** This prologue has 60 problems (as does every problem set in this text), but at this point you have no basis for working the problems in this set. Read, but DO NOT WORK, Problems 11–60.
   a. From this set of problems, name 10 problems that you think you might be able to answer correctly.
   b. From this set of problems, name 10 problems that you know you would not be able to answer correctly.
10. **IN YOUR OWN WORDS** Problems 11–60 are included to give you a quick overview of what this text is about. Select five problems you would like to learn how to solve.

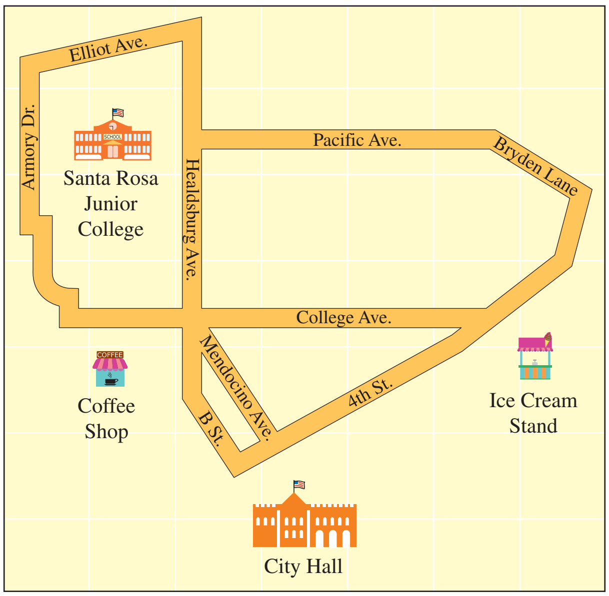
In mathematics, it is important to read the directions before attempting to work any problems. You are NOT expected to be able to work Problems 11–60 at this time. You ARE expected to be able to work those problems AFTER reading this text. So for now, just place each assigned question into one of the following categories:

Yes. I know how to start this problem.
No. I don't think I know how to start this problem.
Hopeful. If someone shows me how to proceed, I think I could answer this question.
Hopeless. I don't think I'll ever be able to answer this question.

11. A long, straight fence having a pole every 8 ft is 1,440 ft long. How many fence poles are needed for the fence? 181
12. How many cards must you draw from a deck of 52 playing cards to be sure that at least two are from the same suit? 5 cards



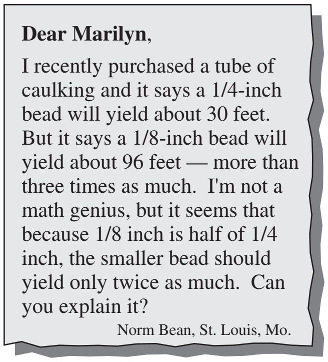
13. How many people must be in a room to be sure that at least four of them have the same birthday (not necessarily the same year)? 1,099
14. Find the units digit of $3^{2015} - 2^{2015}$. 9
15. If a year had two consecutive months with a Friday the thirteenth, which months would they have to be? Feb and Mar
16. On Saturday evenings, a favorite pastime of the high school students in Santa Rosa, California, is to cruise certain streets. The selected routes are shown in the following illustration. Is it possible to choose a route so that all of the permitted streets are traveled exactly once? impossible to trace out this circuit



Santa Rosa street problem

17. What is the largest number that is a divisor of both 210 and 330? 30

18. The news clip shows a letter printed in the "Ask Marilyn" column of *Parade Magazine* (Sept. 27, 1992). How would you answer it?



Dear Marilyn,

I recently purchased a tube of caulking and it says a 1/4-inch bead will yield about 30 feet. But it says a 1/8-inch bead will yield about 96 feet — more than three times as much. I'm not a math genius, but it seems that because 1/8 inch is half of 1/4 inch, the smaller bead should yield only twice as much. Can you explain it?

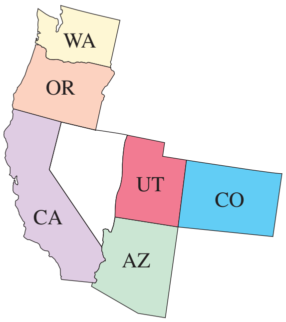
Norm Bean, St. Louis, Mo.

Hint: We won't give you the answer, but we will quote one line from Marilyn's answer: "So the question should be not why the smaller one yields that much, but why it yields that little." Marilyn is correct.

19. If the population of the world on October 12, 2002, was 6.248 billion, when do you think the world population should have reached 7 billion? Calculate the date (to the nearest month) using the information that the world population reached 6 billion on October 12, 1999. March 2011 The world population actually reached 7 billion on October 31, 2011. What does this say about your prediction? Actual growth rate is slower.

20. The Pacific 12 football conference consists of the following schools:

Arizona
Arizona State
Cal Berkeley
Colorado
Oregon
Oregon State
Stanford (CA)
UCLA
USC
Utah
Washington
Washington State



a. Is it possible to visit each of these schools by crossing each common state border exactly once? If so, show the path. yes

b. Is it possible to start the trip in any given state, cross each common state border exactly once, and end the trip in the state in which you started? no

21. If $(a, b) = a \times b + a + b$, what is the value of $((1, 2), (3, 4))$? 119

22. If it is known that all Angelenos are Venusians and all Venusians are Los Angeles residents, then what must necessarily be the conclusion? All Angelenos are Los Angeles residents.

23. If 1 is the first odd number, what is the 473rd odd number? 945

24. If $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, what is the sum of the first 100,000 counting numbers beginning with 1? 5,000,050,000

25. A four-inch cube is painted red on all sides. It is then cut into one-inch cubes. What fraction of all the one-inch cubes are painted on one side only? 3/8

26. If slot machines had two arms and people had one arm, then it is probable that our number system would be based on the digits 0, 1, 2, 3, and 4 only. How would the number we know as 18 be written in such a number system? 33

27. If $M(a, b)$ stands for the larger number in the parentheses, and $m(a, b)$ stands for the lesser number in the parentheses, what is the value of $M(m(1, 2), m(2, 3))$? 2

28. If a group of 50 persons consists of 20 males, 12 children, and 25 women, how many men are in the group? 13

29. There are only five regular polyhedra, and Figure 7 shows the patterns that give those polyhedra. Name the polyhedron obtained from each of the patterns shown.

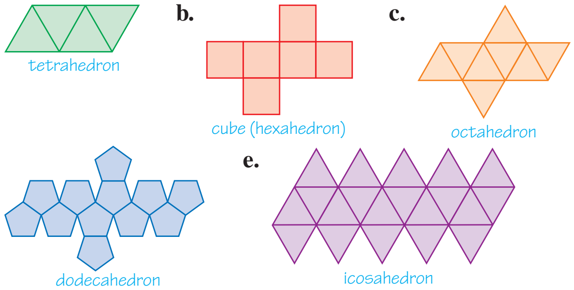


FIGURE 7 **Five regular polyhedra patterns**

30. Jack and Jill decide to exercise together. Jack walks around their favorite lake in 16 minutes and Jill jogs around the lake in 10 minutes. If Jack and Jill start at the same time and at the same place and continue to exercise around the lake until they return to the starting point at the same time, how long will they be exercising? 80 min

31. What is the 1,000th positive integer that is not divisible by 3? 1,499

32. A frugal man allows himself a glass of wine before dinner on every third day, an after-dinner chocolate every fifth day, and a steak dinner once a week. If it happens that he enjoys all three luxuries on March 31, what will be the date of the next steak dinner that is preceded by wine and followed by an after-dinner chocolate? July 14

33. How many trees must be cut to make a trillion one-dollar bills? To answer this question, you need to make some assumptions. Assume that a pound of paper is equal to a pound of wood, and also assume that a dollar bill weighs about one gram. This implies that a pound of wood yields about 450 dollar bills. Furthermore, estimate that an average tree has a height of 50 ft and a diameter of 12 inches. Finally, assume that wood yields about 50 lb/ft^3. more than a million trees

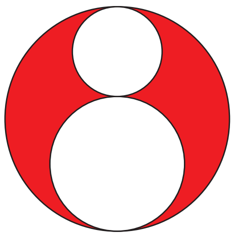
34. Estimate the volume of beer in the six-pack shown in the photograph. about 750,000 gal or almost 8 million standard beer cans

Courtesy Pabst Brewing Company

35. Critique the statement given in the news clip. Sample was not randomly selected.

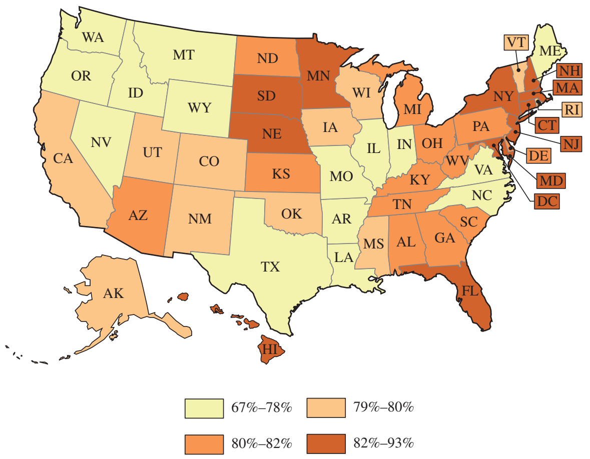
> **Smoking ban**
> Judy Green, owner of the White Restaurant and an adamant opponent of a smoking ban, went so far as to survey numerous restaurants. She cited one restaurant that suffered a 75% decline in business after the smoking ban was activated.

36. The two small circles have radii of 2 and 3. Find the ratio of the area of the smallest circle to the area of the shaded region. 1 to 3



37. A gambler went to the horse races two days in a row. On the first day, she doubled her money and spent \$30. On the second day, she tripled her money and spent \$20, after which she had what she started with the first day. How much did she start with? \$22

38. The map shows the percent of children ages 19–35 months who are immunized by the state. What conclusions can you draw from this map? The percents range from a low of 67% to a high of 93%.



Source: www.cdc.gov/Vaccines/states-surv/his/tables/07/tab03_atigen_state.xls

39. A charter flight has signed up 100 travelers. The travelers are told that if they can sign up an additional 25 persons, they can save \$78 each. What is the cost per person if 100 persons make the trip? \$390

40. Find $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. e

41. Suppose that it costs \$450 to enroll your child in a 10-week summer recreational program. If this cost is prorated (that is, reduced linearly over the 10-week period), express the cost as a function of the number of weeks that have elapsed since the start of the 10-week session. Draw a graph to show the cost at any time for the duration of the session. See IM.

42. Candidates Ramirez (R), Smith (S), and Tillem (T) are running for office. According to public opinion polls, the preferences are (percentages rounded to the nearest percent):

Ranking	38%	29%	24%	10%
1st choice	R	S	T	R
2nd choice	S	R	S	T
3rd choice	T	T	R	S

a. Who will win the plurality vote? Ramirez (R)
b. Who will win the Borda count? Ramirez (R)
c. Does a strategy exist that the voters in the 24% column could use to vote insincerely to keep Ramirez from winning? If the people in the 24% column change to (S, T, R), then Smith will win.

43. Suppose the percentage of alcohol in the blood t hours after consumption is given by

$$C(t) = 0.3e^{-t/2}$$

What is the rate at which the percentage of alcohol is changing with respect to time? $-0.15e^{-t/2}$

44. If a megamile is one million miles and a kilomile is one thousand miles, how many kilomiles are there in 2.376 megamiles? 2,376 kilomiles

45. A map of a small village is shown in Figure 8. To walk from *A* to *B*, Sarah obviously must walk at least 7 blocks (all the blocks are the same length). What is the number of shortest paths from *A* to *B*? 33 paths

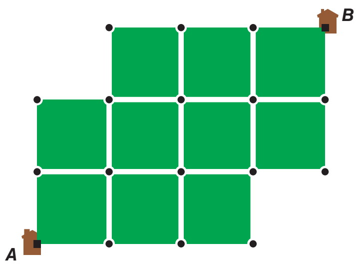


FIGURE 8 **A village map**

46. A hospital wishes to provide for its patients a diet that has a minimum of 100 g of carbohydrates, 60 g of protein, and 40 g of fats per day. These requirements can be met with two foods:

Food	Carbohydrates	Protein	Fats
A	6 g/oz	3 g/oz	1 g/oz
B	2 g/oz	2 g/oz	2 g/oz

It is also important to minimize costs; food A costs \$0.14 per ounce and food B costs \$0.06 per ounce. How many ounces of each food should be bought for each patient per day to meet the minimum daily requirements at the lowest cost? The minimum cost is \$2.52 with 12 g of food A and 14 g of food B.

47. On December 8, 2015, the U.S. national debt hit \$18 trillion and on that date there were 319.4 million people living in the United States. How long would it take to pay off this debt if *every* person paid \$1 per day? About 154 years

48. Find the smallest number of operations needed to build up to the number 100 if you start at 0 and use only two operations: doubling or increasing by 1. *Challenge*: Answer the same question if you want to build up any positive integer n. 9 steps

49. If $\log_2 x + \log_4 x = \log_b x$, what is b? $2^{2/3}$

50. Supply the missing number in the following sequence: 10, 11, 12, 13, 14, 15, 16, 17, 20, 22, 24, __, 100, 121, 10,000. 31

51. How many different configurations can you see in Figure 9? *at least 3*

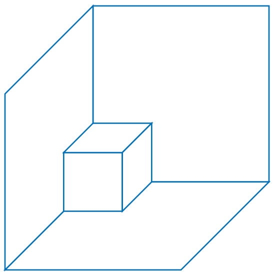


FIGURE 9 **Count the cubes**

52. Answer the question asked in the news clip from the "Ask Marilyn" column of *Parade Magazine* (July 16, 1995). *The first hunter has the best chance.*

Dear Marilyn,

Three safari hunters are captured by a sadistic tribe of natives and forced to participate in a duel to the death. Each is given a pistol and tied to a post the same distance from the other two. They must take turns shooting at each other, one shot per turn. The worst shot of the three hunters (1 in 3 accuracy) must shoot first. The second turn goes to the hunter with 50–50 (1 in 2) accuracy. And (if he's still alive!) the third turn goes to the crack shot (100% accuracy). The rotation continues until only one hunter remains, who is then rewarded with his freedom. Which hunter has the best chance of surviving, and why?

From "Ask Marilyn," by Marilyn vos Savant, *Parade Magazine*, July 16, 1995.

53. Five cards are drawn at random from a pack of cards that have been numbered consecutively from 1 to 104 and have been thoroughly shuffled. What is the probability that the numbers on the cards as they are drawn are in increasing order of magnitude? *1/120*

54. What is the sum of the counting numbers from 1 to 104? *5,460*

55. The Kabbalah is a body of mystical teachings from the Torah. One medieval inscription is shown on the left:

ד	ט	ב
ג	ה	ז
ח	א	ו

4	9	2
3	5	7
8	1	6

The inscription on the left shows Hebrew characters that can be translated into numbers, as shown at the right. What can you say about this pattern of numbers? *The sum of the numbers in every row, column, and diagonal is 15.*

56. What is the maximum number of points of intersection of n distinct lines? *If s_n is the number of intersection points for n lines, then $s_n = s_{n-1} + (n - 1)$.*

57. The equation $P = 153{,}000e^{0.05t}$ represents the population of a city t years after 2000. What is the population of the city in the year 2000? Show a graph of the city's population for the next 20 years. *See IM.*

58. The Egyptians had an interesting, pictorial numeration system. Here is how you would count using Egyptian numerals:

I, II, III, IIII, IIIII, IIIIII, IIIIIII, IIIIIIII, IIIIIIIII, ∩, ∩I, ∩II, ∩III, . . .

Write down your age using Egyptian numerals. The symbol "I" is called a stroke, and "∩" is called a heel bone. The Egyptians used a scroll for 100, a lotus flower for 1,000, a pointing finger for 10,000, a polliwog for 100,000, and an astonished man for the number 1,000,000. *Without* doing any research, write what you think today's date would look like using Egyptian numerals. *Answers vary.*

59. If you start with $1 and double the amount received on the previous day, how much money will you have in 30 days? *$2^{30} - 1$ dollars or more than a billion dollars*

60. Consider two experiments and events defined as follows:

Experiment A: Roll one die 4 times and keep a record of how many times you obtain at least one 6. Event $E = \{$obtain at least one 6 in 4 rolls of a single die$\}$

Experiment B: Roll a pair of dice 24 times and keep a record of how many times you obtain at least one 12. Event $F = \{$obtain at least one 12 in 24 rolls of a pair of dice$\}$

Do you think event E or event F is more likely? You might wish to experiment by rolling dice. *E is more likely.*

THE NATURE OF PROBLEM SOLVING

1

"The idea that aptitude for mathematics is rarer than aptitude for other subjects is merely an illusion which is caused by belated or neglected beginners." —J. F. Herbart

Preview

TOPICS	KEY IDEAS
1.1 Problem Solving	Guidelines for problem solving [1.1]
1.2 Inductive and Deductive Reasoning	Order of operations [1.2]
1.3 Scientific Notation and Estimation	Euler circles [1.2]
	Extended order of operations [1.3]
	Laws of exponents [1.3]
	Inductive vs. deductive reasoning [1.3]

What in the World?

Doug Menuez/Photodisc/Getty Images

"Hey, Tom, what are you taking this semester?" asked Susan. "I'm taking English, history, and math. I can't believe my math teacher," responded Tom. "The first day we were there, she walked in, wrote her name on the board, and then she asked, 'How much space would you have if you, along with everyone else in the world, moved to California?' What a stupid question . . . I would not have enough room to turn around!"

"Oh, I had that math class last semester," said Susan. "It isn't so bad. The whole idea is to give you the ability to solve problems *outside* the class. I want to get a good job when I graduate, and I've read that because of the economy, employers are looking for people with problem-solving skills. I hear that working smarter is more important than working harder."

BOOK REPORTS

Write a 500-word report on one of these books:

Mathematical Magic Show, Martin Gardner (New York: Alfred A. Knopf, 1977).

How to Solve It: A New Aspect of Mathematical Method, George Pólya (New Jersey: Princeton University Press, 1945, 1973).

Chapter Challenge*

$A + B = C$
$A + C = D$
$B + C = E$
$F + H = N$
$G + J = ?$

Q

?

*At the beginning of each chapter, we present a puzzle which represents some pattern. In each case, see if you can fill in the question mark.

Overview

There are many reasons for reading a book, but the best reason is because you want to read it. Although you are probably reading this first page because you were required to do so by your instructor, it is my hope that in a short while you will be reading this text because you *want* to read it. It was written for people who think they don't like mathematics, or people who think they can't work math problems, or people who think they are never going to use math. The common thread in this text is *problem solving*—that is, strengthening your ability to solve problems—not in the classroom, but outside the classroom. This first chapter is designed to introduce you to the nature of problem solving. Notice the first thing you see on this page is the question, "What in the World?" Each chapter begins with such a real-world question that appears later in the chapter. This first one is considered in Problem 59, page 39.

As you begin your trip through this text, I wish you a BON VOYAGE!

A FABLE

Wallnarez/Shutterstock.com

Two young ladies, Shelley and Cindy, came to a town called Mathematics. People had warned them that this was a particularly confusing town. Many people who arrived in Mathematics were very enthusiastic, but could not find their way around, became frustrated, gave up, and left town.

Shelley was strongly determined to succeed. She was going to learn her way through the town. For example, in order to learn how to go from her dorm to class, she concentrated on memorizing this clearly essential information: she had to walk 325 steps south, then 253 steps west, then 129 steps in a diagonal (south-west), and finally 86 steps north. It was not easy to remember all of that, but fortunately she had a very good instructor who helped her to walk this same path 50 times. In order to stick to the strictly necessary information, she ignored much of the beauty along the route, such as the color of the adjacent buildings or the existence of trees, bushes, and nearby flowers. She always walked blindfolded. After repeated exercising, she succeeded in learning her way to class and also to the cafeteria. But she could not learn the way to the grocery store, the bus station, or a nice restaurant; there were just too many routes to memorize. It was so overwhelming! Finally, she gave up and left town; Mathematics was too complicated for her.

Cindy, on the other hand, was of a much less serious nature. To the dismay of her instructor, she did not even intend to memorize the number of steps of her walks. Neither did she use the standard blindfold which students need for learning. She was always curious, looking at the different buildings, trees, bushes, and nearby flowers or anything else not necessarily related to her walk. Sometimes she walked down dead-end alleys in order to find out where they were leading, even if this was obviously superfluous. Curiously, Cindy succeeded in learning how to walk from one place to another. She even found it easy and enjoyed the scenery. She eventually built a building on a vacant lot in the city of Mathematics.*

*My thanks to Emilio Roxin of the University of Rhode Island for the idea for this fable.

1.1 Problem Solving

A Word of Encouragement

Do you think of mathematics as a difficult, foreboding subject that was invented hundreds of years ago? Do you think that you will never be able (or even want) to use mathematics? If you answered "yes" to either of these questions, then I want you to know that I have written this text for you. I have tried to give you some insight into how mathematics is developed and to introduce you to some of the people behind the mathematics. In this text, I will present some of the great ideas of mathematics, and then we will look at how these ideas can be used in an everyday setting to build your problem-solving abilities. *The most important prerequisite for this course is an openness to try out new ideas—a willingness to experience the suggested activities rather than to sit on the sideline as a spectator.* I have attempted to make this material interesting by putting it together differently from the way you might have had mathematics presented in the past. You will find this text difficult if you wait for the text or the teacher to give you answers—instead, *be willing to guess, experiment, estimate, and manipulate*, and try out problems *without fear of being wrong!*

There is a common belief that mathematics is to be pursued only in a clear-cut logical fashion. This belief is perpetuated by the way mathematics is presented in most textbooks. Often it is reduced to a series of definitions, methods to solve various types of problems, and theorems. These theorems are justified by means of proofs and deductive reasoning. I do not mean to minimize the importance of proof in mathematics, for it is the very thing that gives mathematics its strength. But the power of the imagination is every bit as important as the power of deductive reasoning. As the mathematician Augustus De Morgan once said, "The power of mathematical invention is not reasoning but imagination."

Let's begin with a short (nonmathematical) quiz. Write down each answer before you look at the answers in the footnote.

Silly Challenge

1. How long was the Hundred Years' War?

2. Which country makes Panama hats?

3. In which month do the Russians celebrate the October Revolution?

OK, now you are starting to suspect trick questions . . . let's continue.

4. From which animal(s) do we get catgut?

5. The Canary Islands in the Pacific are named after what animal?

6. What is a camel's hair brush made of?

Now, change your goal . . . can you research these questions to obtain one correct answer?

7. What color is a purple finch?

8. Where are Chinese gooseberries from?

9. What was King George VI's first name?

And now, for the bonus question:

10. How long was the Thirty Years' War?*

*Answers to the quiz: **1.** 116 years (1337 to 1453) **2.** Ecuador **3.** November (The Russian calendar at the time was 13 days behind ours.) **4.** From sheep and horses **5.** Dogs (the Latin name is *Insularia Canaria*—Island of the Dogs) **6.** Squirrel fur **7.** Crimson **8.** New Zealand **9.** Albert (Queen Victoria said that no king should ever be called Albert, so he took the name George when he ascended to the throne.) **10.** Thirty years, of course (1618 to 1648)—we included this so that you would get at least one correct answer!

Success in this course is not dependent on what you know or do not know . . . it is based on your willingness to try, to guess, to experiment, to estimate, and to manipulate (we just said that in the opening paragraph). This “Silly Challenge” should be viewed as a quick self-test to see if you are ready to follow this simple rule for success. We continue with other hints for success.

> Mathematics is one component of any plan for liberal education. Mother of all the sciences, it is a builder of the imagination, a weaver of patterns of sheer thought, an intuitive dreamer, a poet. The study of mathematics cannot be replaced by any other activity...

American Mathematical Monthly, Volume 56, 1949, p. 19.

Hints for Success

Mathematics is different from other subjects. One topic builds upon another, and you need to make sure that you understand *each* topic before progressing to the next one.

You must make a commitment to attend each class. Obviously, unforeseen circumstances can come up, but you must plan to attend class regularly. Pay attention to what your teacher says and does, and take notes. If you must miss class, write an outline of the text corresponding to the missed material, including working out each text example on your notebook paper.

You must make a commitment to daily work. Do not expect to save up and do your mathematics work once or twice a week. It will take a daily commitment on your part, and you will find mathematics difficult if you try to “get it done” in spurts. You could not expect to become proficient in tennis, soccer, or playing the piano by practicing once a week, and the same is true of mathematics. Try to schedule a regular time to study mathematics each day.

Read the text carefully. Many students expect to get through a mathematics course by beginning with the homework problems, then reading some examples, and reading the text only as a desperate attempt to find an answer. This procedure is backward; do your homework only *after* reading the text.

Writing Mathematics

The fundamental objective of education has always been to prepare students for life. A measure of your success with this text is a measure of its usefulness to you in your life. What are the basics for your knowledge “in life”? In this information age with access to a world of knowledge on the Internet, we still would respond by saying that the basics remain “reading, ’riting, and ’rithmetic.” As you progress through the material in this text, we will give you opportunities to read mathematics and to consider some of the great ideas in the history of civilization, to develop your problem-solving skills (’rithmetic), and to communicate mathematical ideas to others (’riting). Perhaps you think of mathematics as “working problems” and “getting answers,” but it is so much more. Mathematics is a way of thought that includes all three Rs, and to strengthen your skills you will be asked to communicate your knowledge in written form.

To begin building your skills in writing mathematics, you might keep a journal summarizing each day’s work. Keep a record of your feelings and perceptions about what happened in class. Ask yourself, “How long did the homework take?” “What time of the day or night did I spend working and studying mathematics?” “What is the most important idea that I should remember from the day’s lesson?” To help you with your journals or writing of mathematics, you will find problems in this text designated “**IN YOUR OWN WORDS**.” (For example, look at Problems 1–6 of the problem set at the end of this section.) There are no right answers or wrong answers to this type of problem, but you are encouraged to look at these for ideas of what you might write in your journal.

Journal Ideas

Write in your journal every day.
Include important ideas.
Include new words, ideas, formulas, or concepts.
Include questions that you want to ask later.
If possible, carry your journal with you so you can write in it anytime you get an idea.

Reasons for Keeping a Journal

It will record ideas you might otherwise forget.
It will keep a record of your progress.
If you have trouble later, it may help you diagnose areas for change or improvement.
It will build your writing skills.

Guidelines for Problem Solving

We begin this study of **problem solving** by looking at the *process* of problem solving. As a mathematics teacher, I often hear the comment, "I can do mathematics, but I can't solve word problems." There ***is*** a great fear and avoidance of "real-life" problems because they do not fit into the same mold as the "examples in the book." Few practical problems from everyday life come in the same form as those you study in school.

To compound the difficulty, learning to solve problems takes time. All too often, the mathematics curriculum is so packed with content that the real process of problem solving is slighted and, because of time limitations, becomes an exercise in mimicking the instructor's steps instead of developing into an approach that can be used long after the final examination is over.

Before we build problem-solving skills, it is necessary to build certain prerequisite skills necessary for problem solving. It is my goal to develop your skills in the mechanics of mathematics, in understanding the important concepts, and finally in applying those skills to solve a new type of problem. I have segregated the problems in this text to help you build these different skills:

IN YOUR OWN WORDS This type of problem asks you to discuss or rephrase main ideas or procedures using your own words.

Level 1 Problems These are mechanical and drill problems and are directly related to an example in the text.

Level 2 Problems These problems require an understanding of the concepts and are loosely related to an example in the text.

Level 3 Problems These problems are extensions of the examples but generally do not have corresponding examples.

Problem Solving These require problem-solving skills or original thinking and generally do not have direct examples in the text. These should be considered Level 3 problems.

Research Problems These problems require Internet research or library work. Most are intended for individual research but a few are group research projects. You will find these problems for research in the chapter summary and at the Web address for this text:

www.mathnature.com

The model for problem solving that we will use was first published in 1945 by the great, charismatic mathematician George Pólya. His book *How to Solve It* (Princeton University Press, 1973) has become a classic. In Pólya's book, you will find this

Historical Note

Karl Smith Library

George Pólya (1887–1985)

Born in Hungary, Pólya attended the universities of Budapest, Vienna, Göttingen, and Paris. He was a professor of mathematics at Stanford University. Pólya's research and winning personality earned him a place of honor not only among mathematicians, but among students and teachers as well. His discoveries spanned an impressive range of mathematics, real and complex analysis, probability, combinatorics, number theory, and geometry. Pólya's *How to Solve It* has been translated into 15 languages. His books have a clarity and elegance seldom seen in mathematics, making them a joy to read. For example, here is his explanation of why he became a mathematician: "It is a little shortened but not quite wrong to say: I thought, I am not good enough for physics and I am too good for philosophy. Mathematics is in between."

problem-solving model as well as a treasure trove of strategy, know-how, rules of thumb, good advice, anecdotes, history, and problems at all levels of mathematics. We will refer to his problem-solving model as the **problem-solving procedure,** summarized as follows.

Guidelines for Problem Solving

In order to solve a problem for which there is no immediate solution or no known procedure, use the following steps:

Step 1 *Understand the problem*. Ask questions, experiment, or otherwise rephrase the question in your own words.

Step 2 *Devise a plan.* Find the connection between the data and the unknown. Look for patterns, relate to a previously solved problem or a known formula, or simplify the given information to give you an easier problem.

Step 3 *Carry out the plan.* Check the steps as you go.

Step 4 *Look back*. Examine the solution obtained. In other words, check your answer.

STOP Pay attention to boxes that look like this—they are used to tell you about important procedures that are used throughout the text.

Pólya's original statement of this procedure is reprinted in the following box.*

UNDERSTANDING THE PROBLEM

First

You have to *understand* the problem.

What is the unknown? What are the data? What is the condition? Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce a suitable notation.

Separate the various parts of the condition. Can you write them down?

DEVISING A PLAN

Second

Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found.

Have you seen it before? Or have you seen the same problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Is the problem related to one you have solved before? Could you use it?

Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other? Did you use all the data? Did you see the whole condition? Have you taken into account all essential notions involved in the problem?

CARRYING OUT THE PLAN

Third

Carry out your plan.

Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

LOOKING BACK

Fourth

Examine the solution.

Can you *check the result*? Can you check the argument?

Can you derive the result differently? Can you see it at a glance?

*This is taken word for word as it was written by Pólya in 1941. It was printed in *How to Solve It* (Princeton, NJ: Princeton University Press, 1973).

Let's apply this procedure for problem solving to the map shown in Figure 1.1; we refer to this problem as the **street problem.** Melissa lives at the YWCA (point A) and works at Macy's (point B). She usually walks to work. How many different routes can Melissa take?

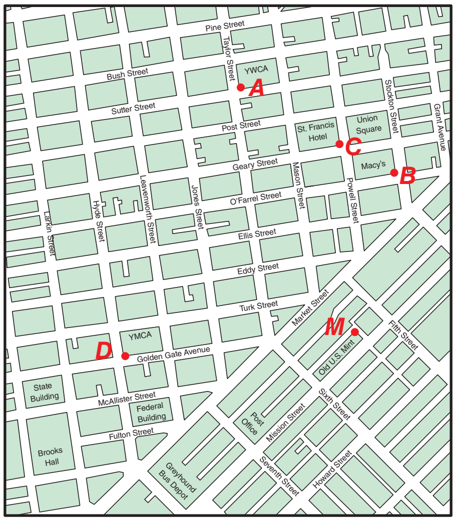


FIGURE 1.1 Portion of a map of San Francisco

Where would you begin with this problem?

Step 1. ***Understand the Problem.*** Can you restate it in your own words? Can you trace out one or two possible paths? What assumptions are reasonable? We assume that Melissa will not do any backtracking—that is, she always travels toward her destination. We also assume that she travels along the city streets—she cannot cut diagonally across a lot or a block.

Step 2. ***Devise a Plan.*** Simplify the question asked. Consider the simplified drawing shown in Figure 1.2.

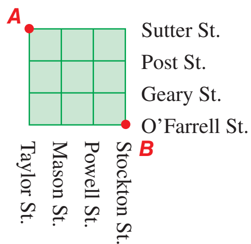


FIGURE 1.2 Simplified portion of Figure 1.1

Step 3. ***Carry Out the Plan.*** Count the number of ways it is possible to arrive at each point, or, as it is sometimes called, a *vertex*.

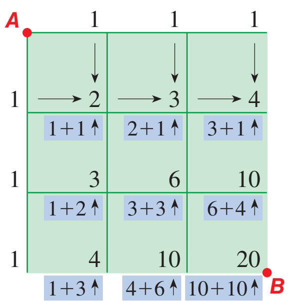
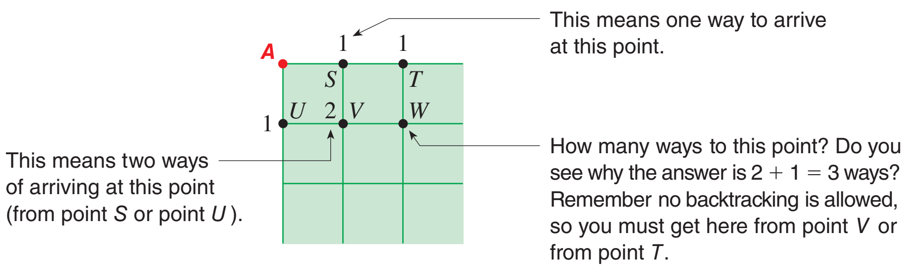
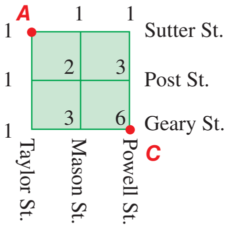


FIGURE 1.3 Map with solution



Now fill in all the possibilities on Figure 1.3, as shown by the above procedure.

Step 4. ***Look Back.*** Does the answer "20 different routes" make sense? Do you think you could fill in all of them?



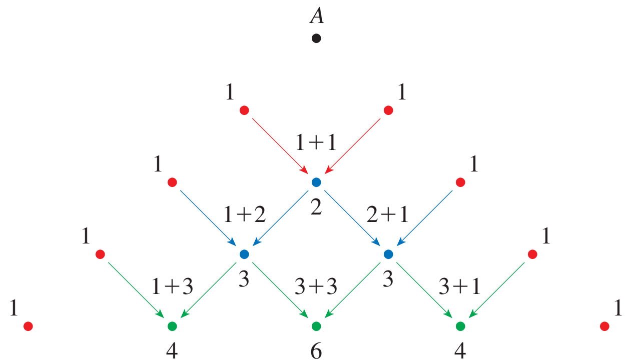
Example 1 Problem solving—from here to there

In how many different ways could Melissa get from the YWCA (point A) to the St. Francis Hotel (point C in Figure 1.1), using the method of Figure 1.3?

Solution Draw a simplified version of Figure 1.3, as shown in the margin. There are 6 different paths.

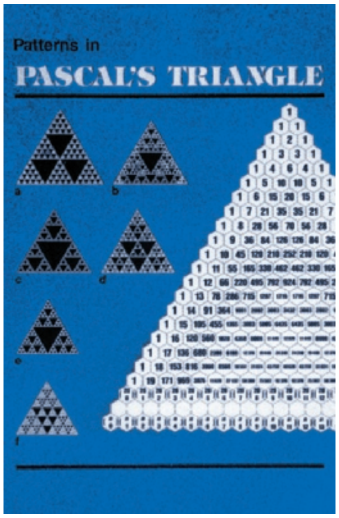
Problem Solving by Patterns

Let's formulate a general solution. Consider a map with a starting point A:



In the Sherlock Holmes mystery *The Final Solution*, Moriarty is a mathematician who wrote a treatise on Pascal's triangle.

Do you see the pattern for building this figure? Each new row is found by adding the two previous numbers, as shown by the arrows. This pattern is known as **Pascal's triangle.** In Figure 1.4, the rows and diagonals are numbered for easy reference.



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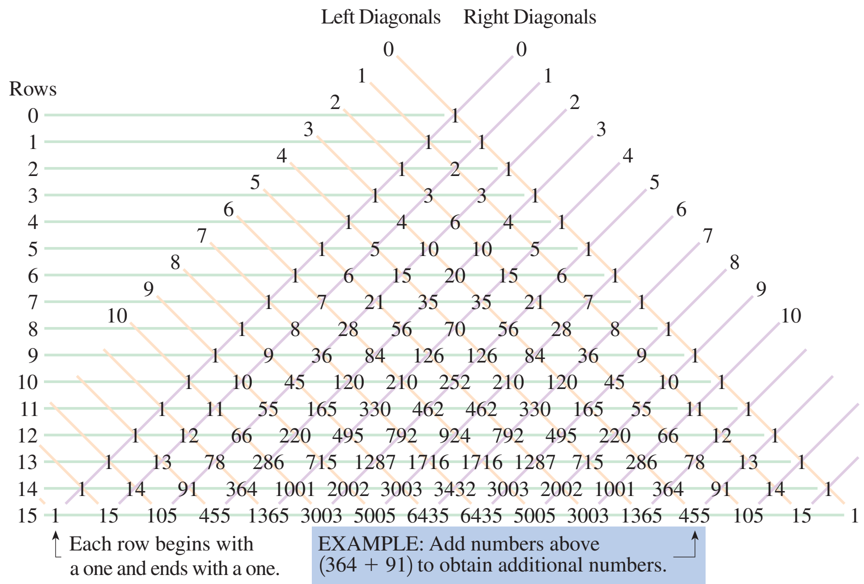


FIGURE 1.4 Pascal's triangle

www.mathnature.com
There is an online interactive version of Pascal's triangle.

How does this pattern apply to Melissa's trip from the YWCA to Macy's? It is 3 blocks down and 3 blocks over. Look at Figure 1.4 and count out these blocks, as shown in Figure 1.5.

Historical Note

Blaise Pascal (1623–1662)

Described as "the greatest 'might-have-been' in the history of mathematics," Pascal was a person of frail health, and because he needed to conserve his energy, he was forbidden to study mathematics. This aroused his curiosity and forced him to acquire most of his knowledge of the subject by himself. At 18, he had invented one of the first calculating machines. However, at 27, because of his health, he promised God that he would abandon mathematics and spend his time in religious study. Three years later he broke this promise and wrote *Traite du triangle arithmétique*, in which he investigated what we today call Pascal's triangle. The very next year he was almost killed when his runaway horse jumped an embankment. He took this to be a sign of God's displeasure with him and again gave up mathematics—this time permanently.

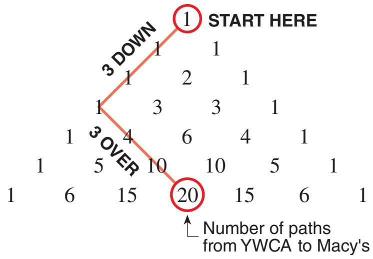
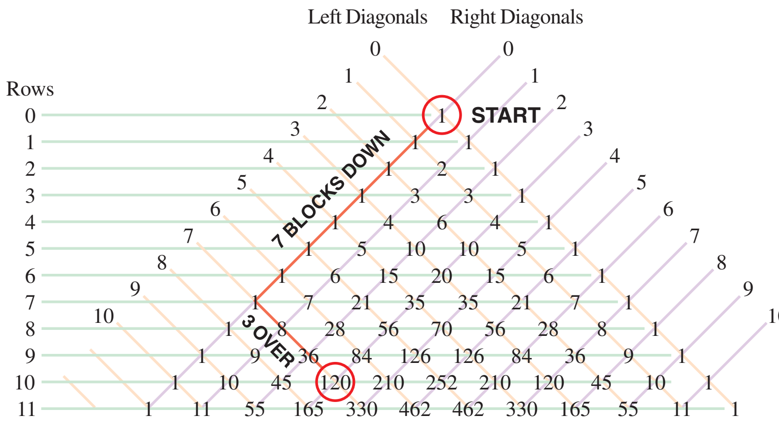


FIGURE 1.5 Using Pascal's triangle to solve the street problem

Example 2 Pascal's triangle to track paths

In how many different ways could Melissa get from the YWCA (point *A* in Figure 1.1) to the YMCA (point *D*)?

Solution Look at Figure 1.1; from point *A* to point *D* is 7 blocks down and 3 blocks left. Use Figure 1.4 as follows:



We see that there are 120 paths.

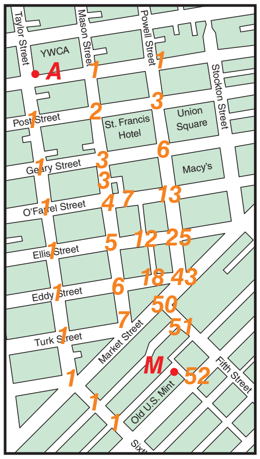
Pascal's triangle applies to the street problem only if the streets are rectangular. If the map shows irregularities (for example, diagonal streets or obstructions), then you must revert back to numbering the vertices.

Example 3 Travel with irregular paths

In how many different ways could Melissa get from the YWCA (point *A*) to the Old U.S. Mint (point *M*)?

Solution If the streets are irregular or if there are obstructions, you cannot use Pascal's triangle, but you can still count the blocks in the same fashion, as shown in the figure in the margin.

There are 52 paths from point *A* to point *M* (if, as usual, we do not allow backtracking).



Problem solving is a difficult task to master, and you are not expected to master it after one section of this text (or even after several chapters of this text). However, you must make building your problem-solving skills an ongoing process. One of the most important aspects of problem solving is to relate new problems to old problems. The problem-solving techniques outlined here should be applied when you are faced with a new problem. When you are faced with a problem similar to one you have already worked, you can apply previously developed techniques (as we did in Examples 1–3). Now, because Example 4 seems to be a new type of problem, we again apply the guidelines for problem solving.